ANALYSIS I EXAMPLES 3

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1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

2. Given $\alpha \in \mathbb{R}$, define $f_\alpha : [-1, 1] \to \mathbb{R}$ by $f_\alpha(x) = |x|^\alpha \sin(1/x)$ for $x \neq 0$ and $f_\alpha(0) = 0$. Is f_0 continuous? Is f_1 differentiable? Draw a table, with 9 columns labelled $\alpha = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ and with 6 rows labelled "f_α bounded", "f_α continuous", "f_α differentiable", "f'_α bounded", "f'_α continuous", "f'_α differentiable". Place ticks and crosses at appropriate places in the table.

3. By applying the mean value theorem to $\log(1 + x)$ on $[0, a/n]$ with $n > |a|$, prove carefully that $(1 + a/n)^n \to e^a$ as $n \to \infty$.

4. Find $\lim_{n\to\infty} n(a^{1/n} - 1)$, where $a > 0$.

5. "Let f' exist on (a, b) and let $c \in (a, b)$. If $c+h \in (a, b)$ then $(f(c+h)-f(c))/h = f'(c + \theta h)$. Let $h \to 0$; then $f'(c + \theta h) \to f'(c)$. Thus f' is continuous at c." Explain why question 2 shows that this argument is false. At what point does it fail?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous and differentiable. Show that f is twice differentiable. Indeed, show that f is infinitely differentiable, and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Comment, in the light of what you know about Taylor series.

7. Find the radius of convergence of each of these power series.

$$\sum_{n\geqslant 0} \frac{2 \cdot 4 \cdot 6 \cdots (2n + 2)}{1 \cdot 4 \cdot 7 \cdots (3n + 1)} z^n \qquad \sum_{n\geqslant 1} \frac{z^{3n}}{n 2^n} \qquad \sum_{n\geqslant 0} \frac{n^n z^n}{n!} \qquad \sum_{n\geqslant 1} n^{\sqrt{n}} z^n$$

8. (L'Hôpital's rule.) Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b). Suppose that $f(a) = g(a) = 0$, that $g'(x)$ does not vanish near a and $f'(x)/g'(x) \to \ell$ as $x \to a$. Show that $f(x)/g(x) \to \ell$ as $x \to a$. Use the rule with $g(x) = x - a$ to show that if $f'(x) \to \ell$ as $x \to a$, then f is differentiable at a with $f'(a) = \ell$.

Find a pair of functions f and g as above for which $\lim_{x\to a} f(x)/g(x)$ exists, but $\lim_{x\to a} f'(x)/g'(x)$ does not.

Investigate the limit as $x \to 1$ of

$$\frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}.$$

9. Find the derivative of $\tan x$. How do you know there is a differentiable inverse function $\tan^{-1} x$ for $x \in \mathbb{R}$? What is its derivative? Now let $g(x) = x - x^3/3 + x^5/5 + \cdots$ for $|x| < 1$. By considering $g'(x)$, explain carefully why $\tan^{-1} x = g(x)$ for $|x| < 1$.

10. The *infinite product* $\prod_{n=1}^{\infty}(1 + a_n)$ is said to *converge* if the sequence $p_n = (1+a_1)\cdots(1+a_n)$ converges. Suppose that $a_n \geqslant 0$ for all n. Putting $s_m = a_1+\cdots+a_m$, prove that $s_n \leqslant p_n \leqslant e^{s_n}$, and deduce that $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Evaluate $\prod_{n=2}^{\infty}(1 + 1/(n^2 - 1))$.

11. Let f be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$. Let $\phi(x) = (f(x) - f(0))/x$ for $x \neq 0$ and $\phi(0) = f'(0)$. Show that ϕ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Using a second order mean value theorem for f, show that $\phi'(x) = f''(\theta x)/2$ for some $0 < \theta < 1$. Hence prove that there exists $c \in (-1, 1)$ with $f''(c) = f(-1) + f(1) - 2f(0)$.

12. Prove the theorem of Darboux: that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f' has the "property of Darboux". (That is to say, if $a < b$ and $f'(a) < z < f'(b)$ then there exists c, $a < c < b$, with $f'(c) = z$.)

13. Using Question 6, construct a function $g : \mathbb{R} \to \mathbb{R}$ that is infinitely-differentiable, positive on a given interval (a, b) and zero elsewhere.

Assuming standard results concerning integration, including the fundamental theorem of calculus, construct a function from $\mathbb{R}$ to $\mathbb{R}$ that is infinitely-differentiable, identically 1 on $[-1, 1]$ and identically 0 outside $(-2, 2)$.

$^{(+)}$Construct an infinitely-differentiable, non-negative, function $\psi : \mathbb{R} \to \mathbb{R}$ that is identically 0 outside $(-2, 2)$, and satisfies

$$\sum_{n=-\infty}^{\infty} \psi(x - n) = 1, \qquad \text{for all } x \in \mathbb{R}.$$